Possible experimental measure theory for the *XXX*-Heisenberg chain

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Raising and lowering operators for the *XXX*-Heisenberg chain are derived explicitly; as a result the dipole moment operator is established. Based on the dipole transition mechanism in an external time-dependent magnetic field, we propose a possible experimental measure theory to detect the energy spectrum of the spin chain. [S1063-651X(99)11208-X]

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I. INTRODUCTION AND MOTIVATION

The *XXX*-Heisenberg chain (HC) defined by

$$H=J\sum_{j=1}^{N}\left(\vec{S}_j\cdot\vec{S}_{j+1}-\frac{1}{4}\right) \tag{1}$$

is certainly one of the most important models in statistical mechanics. The simple form of Eq. (1) belies the rich physical behavior that it displays, and an understanding of the physics of the HC in one-dimension has proved a formidable task for theoretical and mathematical physicists over the last six decades[1–10]. Exact solution of its eigenstates and energy spectrum is given by the Bethe ansatz [11]. In the statement of inverse scattering method, the structure of Bethe ansatz levels is related to the spinon spectrum which is different from the spin-wave theory [12]. Reference [13] has provided an inelastic neutron scattering experiment for the one-dimensional $S=1/2$ Heisenberg antiferromagnet (for $J>0$) $KCuF_3$. However, little recent experimental work was done for the half-odd-integer ferromagnetic (for $J<0$) spin chain. Up to now, to our knowledge, there has not been a mature experimental measure theory that can guide experimental works for the ferromagnetic spin chain. The aim of this paper is to propose a possible experimental measure theory for the ferromagnetic HC in the framework of quantum mechanics, based on the dipole transition mechanism in an external time-dependent magnetic field.

A standard quantum mechanical transition problem in general has the following format. The first quantity we must have is a Hamiltonian $\mathcal{H}$, which can be divided as follows:

$$\mathcal{H}=H_0+H_I, \tag{2}$$

where for some region of coordinate space or time H_I can be neglected. Secondly, when H_I is neglected, it is meaningful to speak of the energy levels and corresponding states of the free Hamiltonian H_0 between which the transitions take place. These transitions are induced by the interaction H_I. Experiments cannot detect directly the stationary energy levels E_n, instead, the frequencies (i.e., the energy intervals) satisfying the Bohr frequency condition (in the unit $\hbar=1$)

$$\omega_{nm}=E_n-E_m. \tag{3}$$

It is stressed that the energy spectrum of H_0 can be determined from experiments is owing to the existence of the external interaction H_I. The physical nature of the external factor, which causes the quantum transition of the microparticles is arbitrary. In particular, it may be the interaction of the microparticles with electromagnetic radiation. Typical examples can be seen in a hydrogen atom or a harmonic oscillator, where a transition from one stationary state to another is realized by an electric dipole moment. The dipole moment operator $\hat{\mathbf{d}}$ of any atom is expressible as a sum of raising and lowering operators $\hat{\mathcal{L}}(n,m)$ between states $|\psi_m\rangle$ and $|\psi_n\rangle$[14]:

$$\hat{\mathbf{d}}=\sum_{n,m}\mathbf{d}_{nm}\hat{\mathcal{L}}(n,m). \tag{4}$$

Usually, in a hydrogen atom or a harmonic oscillator, the dipole moment operator is the coordinate $\mathbf{r}$ or the momentum $\mathbf{p}$ of the particle, and the interaction H_I is expressed by the scalar product of the dipole moment operator and the external field. This kind of dipole transition mechanism is extended to the HC so that its energy spectrum can be detected from experiments.

The paper is organized as follows. In Sec. II, to make this paper self-contained, we briefly review the general definition of raising and lowering operators. In Sec. III, explicit raising and lowering operators for the HC are derived, as a result the dipole moment operator for the HC can be established. In Sec. IV, the interaction H_I is written and experiment detecting the energy spectrum of the ferromagnetic HC is proposed. The discussion is given in the last section.

II. GENERAL DEFINITION OF RAISING AND LOWERING OPERATORS

Operator methods are among basic tools of quantum mechanics. For a physical system described by an observable H, the eigenproblem $H|E\rangle=E|E\rangle$ can be solved exactly due to its raising and lowering operators, without dealing with

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Schödinger equation. In quantum mechanics, the factorization of H into raising and lowering operators for the discrete spectrum is a property of Hilbert space and is not restricted to any particular representation [15]. If H has a discrete spectrum, then it can be written as

$$H=\sum_{n} E_n|\psi_n\rangle\langle\psi_n|, \tag{5}$$

where $|\psi_n\rangle$'s are the complete and orthonormal basis states of H. Thus one way factorization

$$\hat{\mathcal{L}}^{+}\hat{\mathcal{L}}^{-}=H-E_0$$

is provided by operators which have the following spectral decompositions:

$$\hat{\mathcal{L}}^{+}=\sum_{n} (E_{n+1}-E_0)^{1/2}|\psi_{n+1}\rangle\langle\psi_n|,$$

$$\hat{\mathcal{L}}^{-}=\sum_{n} (E_{n+1}-E_0)^{1/2}|\psi_{n}\rangle\langle\psi_{n+1}|. \tag{6}$$

These mutually adjoint operators perform the raising and lowering operators:

$$\hat{\mathcal{L}}^{+}|\psi_n\rangle=(E_{n+1}-E_0)^{1/2}|\psi_{n+1}\rangle,$$

$$\hat{\mathcal{L}}^{-}|\psi_n\rangle=(E_{n}-E_0)^{1/2}|\psi_{n-1}\rangle. \tag{7}$$

From Eq. (5) and Eq. (6), one can have

$$[H,\hat{\mathcal{L}}^{\pm}]=\hat{\mathcal{L}}^{\pm}F^{\pm}, \tag{8}$$

where

$$F^{\pm}=\sum_{n} (E_{n\pm1}-E_n)|\psi_n\rangle\langle\psi_n| \tag{9}$$

is an adjacent energy interval operator, since $F^{\pm}|\psi_n\rangle=(E_{n\pm1}-E_n)|\psi_n\rangle$. [Here we have place $F^{\pm}$ to the right of $\hat{\mathcal{L}}^{\pm}$ in Eq. (8) to allow it to operate directly on eigenfunction $|\psi_n\rangle$, this will simplify the calculations.] In particular, when $F^{\pm}=\pm\hbar\omega$, Eq. (8) corresponds to the usual one in a harmonic oscillator. When $F^{\pm}$ is a function of H, i.e., $F^{\pm}=f^{\pm}(H)$, Eq. (8) becomes

$$[H,\hat{\mathcal{L}}^{\pm}]=\hat{\mathcal{L}}^{\pm}f^{\pm}(H), \tag{10}$$

which is the case shown in Ref. [16]. Equation (8) or Eq. (10) is the general definition of raising and lowering operators expressing by a commutation relation. Note that the explicit form of the raising and lowering operators $\hat{\mathcal{L}}^{\pm}$ for a specific Hamiltonian system need not be mutually adjoint [16].

III. RAISING AND LOWERING OPERATORS FOR THE HC

Although the HC has been studied for such a long time, its raising and lowering operators are not clear yet. The purpose of this section is to derive them explicitly.

Eigenstates with $r(r=0,1,2,\ldots,N)$ down-spins of the HC are given by [11]

$$|\psi_r\rangle=C_r\sum_{m_1<m_2<\cdots<m_r} a(m_1,m_2,\ldots,m_r)\phi(m_1,m_2,\ldots,m_r), \tag{11}$$

where $C_r=[N!/r!(N-r)!]^{-1/2}$ is the normalized constant, $\phi(m_1,m_2,\ldots,m_r)$ represents a spin state with r down-spins on the m_j-th $(j=1,2,\ldots,r)$ sites, the coefficients

$$a(m_1,m_2,\ldots,m_r)=\sum_{P=1}^{r!}\exp\left[i\left(\sum_{j=1}^{r}\theta_{P_j}m_j+\frac{1}{2}\sum \phi_{P_j,P_n}\right)\right], \tag{12}$$

are some exponential functions and defined only for the ordering $m_1<m_2<\ldots<m_r$, and P is any permutation of the r numbers $1,2,\ldots,r$, P_j the number replaced j under this permutation, and $\phi_{jn}=-\phi_{nj}$. In Eq. (11) for the case with $r=0$, $|\psi_0\rangle=|\uparrow\uparrow\ldots\uparrow\rangle$ is the vacuum state with all spins up. $|\psi_r\rangle$ is the simultaneous eigenfunction of H and the z component of the total spin with the eigenvalues:

$$E_r=J\sum_{i=1}^{r}(\cos\theta_i-1),\quad S_z=\frac{N}{2}-r, \tag{13}$$

where θ_i's are related to wave vectors and satisfy the Bethe ansatz equations.

Firstly, let us consider the raising operator that satisfies

$$Q^{+}_{r,r-1}|\psi_{r-1}\rangle=|\psi_r\rangle, \tag{14}$$

namely, the raising operator $Q^{+}_{r,r-1}$ transforms the adjacent eigenstates specified by $r-1$ and r. In general, the $Q^{+}_{r,r-1}$'s are not always the same for different sets $\{r,r-1\}$, but merely have the similar forms. Therefore, when we refer to an operator $Q^{+}_{r,r-1}$, it always acts on $|\psi_{r-1}\rangle$. An arbitrary state $|\psi_r\rangle$ can be obtained by repeated application of $Q^{+}_{r,r-1}$ to the given state $|\psi_0\rangle$ as follows:

$$|\psi_r\rangle=Q^{+}_{r,r-1}Q^{+}_{r-1,r-2}\ldots Q^{+}_{2,1}Q^{+}_{1,0}|\psi_0\rangle. \tag{15}$$

Taking Eq. (14) and $H|\psi_r\rangle=E_r|\psi_r\rangle$ into account, one can verify that

$$([H,Q^{+}_{r,r-1}]=\omega_{r,r-1}Q^{+}_{r,r-1})|\psi_{r-1}\rangle, \tag{16}$$

where $\omega_{r,r-1}$ satisfies Eq. (3). $Q^{+}_{r,r-1}$ will be determined based on Eq. (16) in the following.

Guided by the observation that $-i(\vec{S}_j\times\vec{S}_k)^{-}=S^{-}_jS^{z}_k-S^{-}_kS^{z}_j$ and

$$-i(\vec{S}_j\times\vec{S}_k)^{-}|\overset{j}{\uparrow}\overset{k}{\uparrow}\rangle=\frac{1}{2}(|\overset{j}{\downarrow}\overset{k}{\uparrow}\rangle-|\overset{j}{\uparrow}\overset{k}{\downarrow}\rangle),$$

$$-i(\vec{S}_j\times\vec{S}_k)^{-}|\overset{j}{\downarrow}\overset{k}{\uparrow}\rangle=\frac{1}{2}|\overset{j}{\downarrow}\overset{k}{\downarrow}\rangle, \tag{17}$$

we should expect to obtain that, after introducing the following unified raising operator:

$$Q^{+}_{r,r-1}=-i\sum_{j<k}^{N} W^{(r)}_{jk}(\vec{S}_j\times\vec{S}_k)^{-},\quad W^{(r)}_{jk}=-W^{(r)}_{kj}, \tag{18}$$

Eq. (14) might be satisfied. By the way, it is worth mentioning that the operator $\vec{J}=-i\Sigma_{j<k}^{N}\vec{S}_j\times\vec{S}_k+\Sigma_{j=1}^{N}\nu_j\vec{S}_j$ is a Yangian operator, and the local Yangian operator $-i(\vec{S}_j\times\vec{S}_k)$ make a transition between singlet and triplet states of $\vec{S}_j\cdot\vec{S}_k$ (to the definition of a Yangian and its realizations in quantum mechanics can be seen in [17] and references therein). $Q^{+}_{r,r-1}$ is yielded by combining $W^{(r)}_{jk}$ with the local Yangian operator and summation, hence $Q^{+}_{r,r-1}$ has the most natural form and is a generalization of the Yangian operator. Of course, other forms of raising operators such as $\mathcal{Q}^{+}_{r,r-1}=\Sigma_{j=1}^{N}\alpha_j^{(r)}S_j^{-}$ can also be introduced, we will return to the problem later.

To find the explict form of $Q^{+}_{r,r-1}$, we need to determine the unknown coefficient $W^{(r)}_{jk}$. The direct calculation shows

$$\begin{aligned}[H,Q^{+}_{r,r-1}]=&-J\sum_{j,k=j+1}^{N} W^{(r)}_{j,j+1}\Big\{[(\vec{S}_{j-1}\times\vec{S}_j)\times\vec{S}_{j+1}]^{-}\\&+\frac{1}{2}(S_j^{-}-S_{j+1}^{-})-[\vec{S}_j\times(\vec{S}_{j+1}\times\vec{S}_{j+2})]^{-}\Big\}\\&-J\sum_{j,k\geqslant j+2}^{N} W^{(r)}_{j,k}\{[(\vec{S}_{j-1}\times\vec{S}_j)\times\vec{S}_k]\}^{-}-[(\vec{S}_j\times\vec{S}_{j+1})\times\vec{S}_k]^{-}\\&+[\vec{S}_j\times(\vec{S}_{k-1}\times\vec{S}_k)]^{-}-[\vec{S}_j\times(\vec{S}_k\times\vec{S}_{k+1})]^{-}\}.\end{aligned} \tag{19}$$

In the following, only $|\psi_0\rangle$ is presumed known, and now we consider the cases $r=1,2,\ldots$ successively.

(a) $r=1$. After acting Eq. (19) on $|\psi_0\rangle$, one obtains

$$\begin{aligned}[H,Q^{+}_{1,0}]|\psi_0\rangle=&-iJ\sum_{j,k=j+1}^{N}\left[\left(\frac{W^{(1)}_{j+1,j+2}+W^{(1)}_{j-1,j}}{2W^{(1)}_{j,j+1}}-1\right)W^{(1)}_{j,j+1}(\vec{S}_j\times\vec{S}_{j+1})^{-}\right]|\psi_0\rangle\\&+iJ\sum_{j,k=j+1}^{N}[(W^{(1)}_{j+1,j+2}S^{-}_{j+2}+W^{(1)}_{j-1,j}S^{-}_{j-1})(\vec{S}_j\times\vec{S}_{j+1})^{z}]|\psi_0\rangle\\&-iJ\sum_{j,k\geqslant j+2}^{N}\left[\left(\frac{W^{(1)}_{j+1,k+1}+W^{(1)}_{j-1,k-1}}{2W^{(1)}_{j,k}}-1\right)W^{(1)}_{jk}(\vec{S}_j\times\vec{S}_k)^{-}\right]|\psi_0\rangle\\&+iJ\sum_{j,k\geqslant j+2}^{N} W^{(1)}_{jk}[S_k^{-}(\vec{S}_{j-1}\times\vec{S}_j)^{z}-S_k^{-}(\vec{S}_j\times\vec{S}_{j+1})^{z}\\&-S_j^{-}(\vec{S}_{k-1}\times\vec{S}_k)^{z}+S_j^{-}(\vec{S}_k\times\vec{S}_{k+1})^{z}]|\psi_0\rangle.\end{aligned} \tag{20}$$

Since $(\vec{S}_j\times\vec{S}_k)^{z}|\psi_0\rangle=0$, then Eq. (20) becomes

$$[H,Q^{+}_{1,0}]|\psi_0\rangle=-iJ\sum_{j<k}^{N}\left[\left(\frac{W^{(1)}_{j+1,k+1}+W^{(1)}_{j-1,k-1}}{2W^{(1)}_{jk}}-1\right)W^{(1)}_{jk}(\vec{S}_j\times\vec{S}_k)^{-}\right]|\psi_0\rangle. \tag{21}$$

Comparing Eq. (21) with Eq. (16), $Q^{+}_{1,0}$ is a raising operator unless the factor

$$\frac{W^{(1)}_{j+1,k+1}+W^{(1)}_{j-1,k-1}}{2W^{(1)}_{jk}}$$

is a real number and does not depend on j and k. If set

$$W^{(1)}_{jk}=\frac{C_1}{C_0}\frac{2}{N}[a(j)-a(k)],\quad a(j)=\exp(ij\theta), \tag{22}$$

Eq. (21) yields

$$[H,Q^{+}_{1,0}]|\psi_0\rangle=(\omega_{10}Q^{+}_{1,0})|\psi_0\rangle, \tag{23}$$

with

$$\omega_{10}=J\left(\frac{W^{(1)}_{j+1,k+1}+W^{(1)}_{j-1,k-1}}{2W^{(1)}_{jk}}-1\right)=J(\cos\theta-1) \tag{24}$$

is the energy interval between E_1 and E_0. Under periodic boundary condition $a(m+N)=a(m)$, it is well-known that

$$\theta=\frac{2\pi}{N}n;\quad n=\pm1,\dots,\pm\left(\frac{N}{2}-1\right),\pm\frac{N}{2}.$$

and $\Sigma_{m=1}^{N}a(m)=0$. Since $E_0=0$, Eq. (24) leads to $E_1=J(\cos\theta-1)$. After acting $Q^{+}_{1,0}$ on $|\psi_0\rangle$, the next wave function $|\psi_1\rangle$ is obtained.

(b) $r=2$. We set

$$|\psi_2\rangle=\sum_{j<k}a(j,k)\phi(j,k) \tag{25}$$

with unknown expansion coefficient $a(j,k)$. The direct calculation shows

$$\begin{aligned}[H,Q^{+}_{2,1}]|\psi_1\rangle=&J\sum_{j,k=j+1}^{N}\left[\frac{a(j-1,j+1)+a(j,j+2)}{2a(j,j+1)}-2\right]a(j,j+1)\phi(j,j+1)\\&+J\sum_{j,k\geqslant j+2}^{N}\left[\frac{a(j-1,k)+a(j+1,k)+a(j,k-1)+a(j,k+1)}{2a(j,k)}-2\right]a(j,k)\phi(j,k)\\&-E_1(Q^{+}_{2,1}|\psi_1\rangle)-iJ\sum_{j<k}^{N}\left\{\left[\sum_{j=1}^{N}\left(\vec{S}_j\cdot\vec{S}_{j+1}-\frac{1}{4}\right)\right](\vec{S}_j\times\vec{S}_k)^{-}W^{(2)}_{j,k}\sum_{m\neq j,k}^{N}a(m)\phi(m)\right\}.\end{aligned} \tag{26}$$

To make $Q^{+}_{2,1}$ a raising operator of H, we must require

$$Q^{+}_{2,1}|\psi_1\rangle=|\psi_2\rangle=\sum_{j<k}a(j,k)\phi(j,k), \tag{27}$$

$$W^{(2)}_{jk}\sum_{m\neq j,k}^{N}a(m)\phi(m)=0, \tag{28}$$

and

$$E_2=\frac{a(j-1,k)+a(j+1,k)+a(j,k-1)+a(j,k+1)}{2a(j,k)}-2=\frac{a(j-1,j+1)+a(j,j+2)}{2a(j,j+1)}-2. \tag{29}$$

Since $a(m)=\exp(im\theta)$ is an exponential function, then one can find that

$$\frac{1}{i\theta}\frac{\partial}{\partial m}a(j)=\delta_{jm}a(m)\quad(m,j=1,2,\dots,N), \tag{30}$$

where $m=x_m$ is understood as the coordinate of the spin located on the mth site of the lattice.

Due to Eq. (30), to make Eq. (28) be valid, one finds $W^{(2)}_{jk}$ can be the following solution:

$$W^{(2)}_{jk}=\frac{C_2}{C_1}\frac{1}{i\theta}a(j,k)\left[\frac{1}{a(j)}\frac{\partial}{\partial(m=j)}-\frac{1}{a(k)}\frac{\partial}{\partial(m=k)}\right]\quad(j<k), \tag{31}$$

with a still unknown coefficient $a(j,k)$. $a(j,k)$ will be determined by requiring that the two factors of the right-hand side of Eq. (29) are real numbers and do not depend on j and k. Obviously, if we choose $a(j,k)$ to be the usual result of the Bethe ansatz, i.e.,

$$a(j,k)=Ce^{ij\theta_1}e^{ik\theta_2}+C'e^{ij\theta_2}e^{ik\theta_1}\quad(j,k=1,2,\dots,N), \tag{32}$$

it yields

$$\frac{a(j-1,k)+a(j+1,k)+a(j,k-1)+a(j,k+1)}{2a(j,k)}=\cos\theta_1+\cos\theta_2 .$$

From Eq. (29) we must require

$$\frac{a(j-1,j+1)+a(j,j+2)}{2a(j,j+1)}=\cos\theta_1+\cos\theta_2, \tag{33}$$

thus

$$\frac{C}{C'}=-\frac{1-2e^{i\theta_1}+e^{i(\theta_1+\theta_2)}}{1-2e^{i\theta_2}+e^{i(\theta_1+\theta_2)}}. \tag{34}$$

On the other hand, from the periodic condition $a(j,k)=a(k,j+N)$, we obtain

$$e^{iN\theta_1}=\frac{C}{C'}=e^{-iN\theta_2}. \tag{35}$$

Equations (34) and (35) lead to the usual Bethe ansatz equations

$$N\theta_1=2\pi I-\Theta(\theta_1,\theta_2),\quad N\theta_2=2\pi I'-\Theta(\theta_2,\theta_1), \tag{36}$$

with

$$\Theta(\theta,\theta')=2\arctan\left\{\frac{\sin\dfrac{\theta-\theta'}{2}}{\cos\dfrac{\theta+\theta'}{2}-\cos\dfrac{\theta-\theta'}{2}}\right\}$$

$$=2\arctan\left\{\frac{1}{2}\left[\cot\frac{\theta}{2}-\cot\frac{\theta'}{2}\right]\right\} \tag{37}$$

is an odd function, i.e., $\Theta(\theta,\theta')=-\Theta(\theta',\theta)$, I and I' $(I'\neq I)$ belong to the set $\{\pm\frac{1}{2},\pm\frac{3}{2},\ldots,\pm(N-1)/2\}$. Thus the energy E_2 is recovered for the case with $r=2$ of Eq. (13) naturally. Since $a(j,k)$ is defined in the ordering $j<k$, to make $W^{(2)}_{jk}=-W^{(2)}_{kj}$, we rewrite $W^{(2)}_{jk}$ as

$$W^{(2)}_{jk}=\frac{C_2}{C_1}\frac{1}{i\theta}A(j,k)\left[\frac{1}{a(j)}\frac{\partial}{\partial(m=j)}-\frac{1}{a(k)}\frac{\partial}{\partial(m=k)}\right], \tag{38}$$

with

$$A(j,k)=A(k,j)=\begin{cases}a(j,k) & \text{if } j<k,\\ a(k,j) & \text{if } j>k\end{cases}. \tag{39}$$

One can note that $W^{(1)}_{j,k}$ is a number; $W^{(2)}_{j,k}$ is a partial differential operator acting on the coefficients $a(m)$'s. From Eq. (30), one can obtain

$$\frac{1}{i\theta}\frac{\partial}{\partial m}|\psi_1\rangle=\frac{1}{i\theta}\frac{\partial}{\partial m}\sum_{m=1}^{N}a(m)\phi(m)=a(m)\phi(m), \tag{40}$$

hence, the action of the partial differential operator $(1/i\theta)\times(\partial/\partial m)$ is clear, when it acts on $|\psi_1\rangle$, the term $a(m)\phi(m)$ is picked up among $|\psi_1\rangle$. Owing to these, direct calculation shows that Eq. (27) is valid naturally.

(c) For general r, making use of the similar analysis, one can obtain $Q^+_{r,r-1}$ with the general solution

$$W^{(r)}_{j,k}=\frac{C_r}{C_{r-1}}\frac{2}{r}\frac{1}{i\theta_1}\frac{1}{i\theta_2}\cdots\frac{1}{i\theta_{r-1}}$$
$$\times\left\{\sum_{l_1,l_2,\ldots,l_{r-2}\neq j,k}^{N}A(j,k,l_1,l_2,\ldots,l_{r-2})\right.$$
$$\times\left[\frac{1}{A(j,l_1,l_2,\ldots,l_{r-2})}\frac{\partial}{\partial j}\right.$$
$$\left.\left.-\frac{1}{A(k,l_1,l_2,\ldots,l_{r-2})}\frac{\partial}{\partial k}\right]\frac{\partial}{\partial l_1}\frac{\partial}{\partial l_2}\cdots\frac{\partial}{\partial l_{r-2}}\right\}, \tag{41}$$

and the coefficient $A(j,k,l_1,l_2,\ldots,l_{r-2})$ has the similar meaning as $A(j,k)$ shown in Eq. (39). Simultaneously, the action of the partial differential operator $(1/i\theta_1)\times(1/i\theta_2)\cdots(1/i\theta_r)(\partial/\partial m_1)(\partial/\partial m_2)\cdots(\partial/\partial m_r)$ is picking up the term $a(m_1,m_2,\ldots,m_r)\phi(m_1,m_2,\ldots,m_r)$ among $|\psi_r\rangle$.

Now we return to the question whether the raising operator $Q^+_{r,r-1}$ can take other forms. Careful calculations show that

$$-i\sum_{j<k}^{N}W^{(r)}_{jk}(\vec{S}_j\times\vec{S}_k)^-|\psi_{r-1}\rangle=\sum_{j=1}^{N}\alpha^{(r)}_jS^-_j|\psi_{r-1}\rangle=|\psi_r\rangle, \tag{42}$$

where

$$\alpha^{(r)}_j=\frac{1}{2}\sum_{k\neq j}^{N}W^{(r)}_{jk}, \tag{43}$$

i.e., when acts on $|\psi_{r-1}\rangle$, $Q^+_{r,r-1}$ can be simplified to $\mathcal{Q}^+_{r,r-1}=\Sigma_{j=1}^{N}\alpha^{(r)}_jS^-_j$, whose form is more simple. However, $\alpha^{(r)}_j$ is more complicate than $W^{(r)}_{jk}$, hence is more difficult to determine than $W^{(r)}_{jk}$. It is the reason why we introduce $Q^+_{r,r-1}$ in the beginning, but not $\mathcal{Q}^+_{r,r-1}$. Equation (42) means that $\mathcal{Q}^+_{r,r-1}$ has the same effect as $Q^+_{r,r-1}$, they are both the raising operators.

Next, we consider the lowering operator $Q^-_{r-1,r}$. Guided by the observation $i(\vec{S}_j\times\vec{S}_k)^+=S^+_jS^z_k-S^+_kS^z_j$ and

$$\left[i(\vec{S}_j\times\vec{S}_k)^++\frac{1}{2}(S^+_j+S^+_k)\right]|\overset{j}{\downarrow}\overset{k}{\downarrow}\rangle=|\overset{j}{\downarrow}\overset{k}{\uparrow}\rangle,$$

$$(\vec{S}_j\times\vec{S}_k)^+|\overset{j}{\downarrow}\overset{k}{\uparrow}\rangle=\frac{1}{2}|\overset{j}{\uparrow}\overset{k}{\uparrow}\rangle, \tag{44}$$

we set

$$Q^{-}_{r-1,r}=i\sum_{j<k}^{N} W'^{(r)}_{jk}(\vec{S}_j\times\vec{S}_k)^{+}+\sum_{j=1}^{N}\beta_j^{(r)}S_j^{+},$$

$$W'^{(r)}_{jk}=-W'^{(r)}_{kj}, \tag{45}$$

which is different from the raising operator by a translation term $\Sigma_{j=1}^{N}\beta_j^{(r)}S_j^{+}$.

Similarly, from the definition

$$([H,Q^{-}_{r-1,r}]=\omega_{r-1,r}Q^{-}_{r-1,r})\ \ |\psi_r\rangle, \tag{46}$$

one has (i) for $r=1$,

$$W'^{(1)}_{jk}=\frac{C_0}{C_1}\frac{2}{N(N-1)}\frac{1}{i\theta}\left[\frac{1}{a(j)}\frac{\partial}{\partial(m=j)}-\frac{1}{a(k)}\frac{\partial}{\partial(m=k)}\right],\quad \beta_j^{(1)}=0, \tag{47}$$

(ii) for $r=2$,

$$W'^{(2)}_{jk}=\frac{C_1}{C_2}\frac{1}{N-1}\frac{a(j)-a(k)}{A(j,k)}\frac{1}{i\theta_1}\frac{1}{i\theta_2}\frac{\partial}{\partial j}\frac{\partial}{\partial k},$$

$$\beta_j^{(2)}=\frac{C_1}{C_2}\frac{1}{N-1}\frac{1}{i\theta_1}\frac{1}{i\theta_2}\sum_{m\neq j}^{N}\frac{a(j)+a(m)}{A(j,m)}\frac{\partial}{\partial m}\frac{\partial}{\partial j}, \tag{48}$$

and (iii) for general $r>2$,

$$W'^{(r)}_{jk}=\frac{C_{r-1}}{C_r}\frac{1}{N-r+1}\frac{1}{i\theta_1}\frac{1}{i\theta_2}\cdots\frac{1}{i\theta_r}\sum_{l_1,\dots,l_{r-2}\neq j,k}\times\left[\frac{A(j,l_1,\dots,l_{r-2})-A(k,l_1,\dots,l_{r-2})}{A(j,k,l_1,\dots,l_{r-2})}\frac{\partial}{\partial l_1}\cdots\frac{\partial}{\partial l_{r-2}}\right]\frac{\partial}{\partial j}\frac{\partial}{\partial k},$$

$$\beta_j^{(r)}=\frac{C_{r-1}}{C_r}\frac{1}{N-r+1}\frac{1}{i\theta_1}\frac{1}{i\theta_2}\cdots\frac{1}{i\theta_r}\times\left\{\sum_{m_1,\cdots,m_{r-1}\neq j}\frac{1}{A(j,m_1,m_2,\cdots,m_{r-1})}\left[\sum_{(l_1,\dots,l_{r-2})\in(m_1,\dots,m_{r-1})}A(j,l_1,\dots,l_{r-2})+A(m_1,m_2,\dots,m_{r-1})\right]\frac{\partial}{\partial m_1}\frac{\partial}{\partial m_2}\cdots\frac{\partial}{\partial m_{r-1}}\right\}\frac{\partial}{\partial j}. \tag{49}$$

Like $Q^{+}_{r,r-1}$, when acting on $|\psi_r\rangle$, $Q^{-}_{r-1,r}$ can be simplified to $\mathcal{Q}^{-}_{r-1,r}=\Sigma_{j=1}^{N}(\alpha'^{(r)}_j S_j^{-})$, with

$$\alpha'^{(r)}_j=\frac{1}{2}\sum_{k\neq j}^{N}W'^{(r)}_{j,k}+\beta_j^{(r)}. \tag{50}$$

Consequently, the lowering operators $Q^{-}_{r-1,r}$ or $\mathcal{Q}^{-}_{r-1,r}$ are also found.

In particular, $\mathcal{Q}^{+}_{1,0}=\Sigma_{m=1}^{N}a(m)S_m^{-}$, $\mathcal{Q}^{-}_{0,1}$ can be simplified to a more simple form $\mathcal{Q}^{-}_{0,1}=\Sigma_{m=1}^{N}a^{-1}(m)S_m^{+}$ when it acts on $|\psi_1\rangle$. These two operators are mutually adjoint. However, for general $r>2$, the Hermitian properties for $\mathcal{Q}^{+}_{r,r-r}$ and $\mathcal{Q}^{-}_{r-1,r}$ are not held.

IV. INTERACTION H_I AND EXPERIMENTAL MEASUREMENT

The Hamiltonian of the HC shown in Eq. (1) is regarded as the free Hamiltonian H_o. To write the interaction H_I, we have to write the dipole moment operator at first. From Eq. (6), one obtains

$$\hat{\mathcal{L}}^{+}(r+1,r)=\sum_r[(E_{r+1}-E_0)^{1/2}Q^{+}_{r+1,r}|\psi_r\rangle\langle\psi_r|],$$

$$\hat{\mathcal{L}}^{-}(r-1,r)=\sum_r[(E_r-E_0)^{1/2}Q^{-}_{r-1,r}|\psi_r\rangle\langle\psi_r|]. \tag{51}$$

Here $Q^{\pm}$ can be replaced by $\mathcal{Q}^{\pm}$. Furthermore

$$\hat{\mathcal{L}}^{+}(n,m)=[\hat{\mathcal{L}}^{+}(r+1,r)]^{n-m},$$

$$\hat{\mathcal{L}}^{-}(m,n)=[\hat{\mathcal{L}}^{-}(r-1,r)]^{n-m}\quad(n>m) \tag{52}$$

from Eq. (4) the dipole moment operator is established.

Since the HC model is a spin-spin coupling system, it can be affected by an external magnetic field. With the help of the dipole moment operator, the interaction Hamiltonian for the HC in a time-dependent magnetic field $\mathbf{B}(t)$ can be expressed as

$$H_I(t)=\hat{\mathbf{d}}\cdot\mathbf{B}(t), \tag{53}$$

where

$$\mathbf{B}(t)=\sum_\lambda[\mathbf{e}(\lambda)\mathcal{E}_\lambda^{(+)}(t)+\mathbf{e}^*(\lambda)\mathcal{E}_\lambda^{(-)}(t)], \tag{54}$$

and

$\mathcal{E}_\lambda^{(+)}(t)=\mathcal{E}_\lambda(t)\exp(-i\omega_\lambda t), \quad \mathcal{E}_\lambda^{(-)}(t)=\mathcal{E}_\lambda^*(t)\exp(+i\omega_\lambda t).$

For the field $\mathbf{B}(t)=\mathbf{B}_0+\mathbf{B}_1(t)$, we propose to put a static magnetic field $\mathbf{B}_0$ in the z direction at first, then to a ferromagnetic HC, in low temperature, its ground state is the vacuum state $|\psi_0\rangle$ with all spins up. Secondly, we superpose on the static field a time-dependent magnetic field $\mathbf{B}_1(t)$ which is perpendicular to $\mathbf{B}_0$. Hence, the time-dependent Schrödinger equation is

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle=(H+\hat{\mathbf{d}}\cdot\mathbf{B}(t))|\Psi(t)\rangle. \tag{55}$$

The general state $|\Psi(t)\rangle$ is written as an expansion

$$|\Psi(t)\rangle=\sum_{n=1} C_n(t)|\psi_n\rangle\exp(-iE_n t), \tag{56}$$

where $|\psi_n\rangle$'s are eigenstates of H, the Hamiltonian of the HC. We thereby obtain a set of coupled equations

$$-i\frac{d}{dt}C_n(t)=\sum_k\sum_\lambda \mathbf{d}_{nk}\cdot\mathbf{e}(\lambda)\mathcal{E}_\lambda \exp[-i(\omega_\lambda-\omega_{nk})t]C_k(t) +\sum_k\sum_\lambda \mathbf{d}_{nk}\cdot\mathbf{e}^*(\lambda)\mathcal{E}_\lambda^* \exp[+i(\omega_\lambda -\omega_{nk})t]C_k(t), \tag{57}$$

with $\omega_{nk}=E_n-E_k$ is the Bohr transition frequency. The first and the second excitations should be states with one and two spins down, i.e., $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively. Obviously, when $\omega_\lambda=\omega_{nk}$, the magnetic resonance phenomena would happen, thus the energy intervals of the spin chain can be detected.

V. CONCLUSION AND DISCUSSION

In our work, based on the dipole transition mechanism in an external time-dependent magnetic field, a possible experimental measure theory to detect the energy spectrum of the ferromagnetic HC is proposed. Of course, there can be other possible mechanisms. A question may arise naturally: How does such a measure theory work for an antiferromagnetic HC? In the following, we would like to make some discussions restricting to the dipole transition mechanism.

(i) Starting from the ferromagnetic ''vacuum'' state $|\psi_0\rangle$, if the raising operators are acted for enough times, then for even-spins antiferromagnetic HC, it will reach the ground state. The corresponding ground state energy was first calculated by Hulthén using Bethe's method [18]. The ground state is a singlet with total spin $S_T=0$ (for $N=$even integer), therefore the number of spin-deviates in the ground state is $r=N/2$ (the proof that the total spin is indeed minimal in the ground state is found in [19]). des Cloiseaux and Pearson (dCP) were the first to study the elementary excitations [20], which they interpreted as spin-wave-like states with $S_T=1$. It was later shown by Faddeev and Takhtajan [21] that the natural excitations (spinons) actually have $S_T=1/2$, and hence fermions. The underlying excitations occur only in pairs [21]. The dCP states are now understood to be a superposition of two spinons, one of which carries zero momentum. In our experimental measure theory, if the energy spectrum is detected by the dipole transition mechanism, people might ask: What are the first and the second excited states? And what are their degeneracies? These problems are still open and under investigation. However, guided by the observation that the electric dipole moment usually transforms a stationary state to its adjacent states in a harmonic oscillator or a hydrogen atom, it might guess that one and two spin-deviates to the ground state of the antiferromagnetic HC should correspond to the first and the second excited states, respectively. The $r>2$ excitations would not be simple compounds of $r=1$ and $r=2$, since the antiferromagnetic HC is a strong spin-spin coupling model, nonlinear excitations should play an important role in the dynamical behavior of the one-dimensional system.

(ii) Eventually, we would like to present an illuminating (but not a mathematically rigorous) argument on the low-temperature thermodynamic properties. In the measure theory described above, the Bohr transition frequency ω_{nk} equals to the magnetic resonance frequency ω_λ, which is carried by a photon. The occupation number should be

$$\langle n\rangle=\frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{k_B T}\right)-1},$$

where k_B is the Boltzmann constant. If the dispersion relation $\omega(k)$ is linear in k, one can show that (in one dimension)

$$\lim_{T\to 0} c(T)=\frac{\partial}{\partial T}\int_0^\infty \langle n\rangle dk \propto \frac{\pi^2 k_B^2}{3\hbar}\,T,$$

i.e., the specific heat is proportional to T in the low temperature.

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